

Directed Subdifferentiable Functions and the Directed Subdifferential Without DC Structure

Robert Baier*, Elza Farkhi[†] and Vera Roshchina[‡]

December 10, 2012

Abstract

We show that the directed subdifferential introduced for differences of convex (DC) functions by Baier and Farkhi can be constructed from the directional derivative without using any information on the DC structure of the function. The new definition extends to a more general class of functions, which includes Lipschitz functions definable on o-minimal structure and quasidifferentiable functions.

Keywords: nonconvex subdifferentials, directional derivatives, difference of convex (DC) functions, differences of sets

MSC: 49J52, 90C26, 26B25, 58C20

1 Introduction

The directed subdifferential was originally introduced in [3] for DC (difference of convex) functions as the difference of the Moreau-Rockafellar subdifferentials of two convex functions embedded in the space of directed sets [1, 2].

*Chair of Applied Mathematics, University of Bayreuth, 95440 Bayreuth, Germany, robert.baier@uni-bayreuth.de

[†]School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, 69978 Tel Aviv, Israel, elza@post.tau.ac.il

[‡]Collaborative Research Network, University of Ballarat, F Building, Mount Helen Campus, PO BOX 663, Ballarat VIC 3353, Australia, vroshchina@ballarat.edu.au

Furthermore, as the directional derivative of any quasidifferentiable (QD) function is the difference of the support functions of two convex compacts, the directed subdifferential was extended in [5, 6] to the class of quasidifferentiable (QD) functions.

The directed subdifferential enjoys exact calculus rules, and gives the same sharp optimality conditions as the Demyanov-Rubinov quasidifferential or the Dini subdifferential and superdifferential. In contrast to the quasidifferential which is an equivalence class of pairs of convex compact sets, the directed subdifferential, as a uniquely defined difference, does not depend on a chosen representing pair, which eliminates the ‘inflation’ in size when calculus rules are applied repeatedly. Moreover, one of the major challenges in quasidifferential calculus associated with the DC decomposition is not relevant for the directed subdifferential: in this work we show that the directed subdifferential can be constructed without any knowledge of the underlying DC structure. In fact, the function (or its directional derivative) does not even need to be DC.

Here we extend the domain of definition of the directed subdifferential to a rather large class of functions on \mathbb{R}^n , essentially wider than QD functions. This class of functions includes all locally Lipschitz functions definable on o-minimal structure (see [10, 11, 15]). To our best knowledge, whether or not Lipschitz tame functions are DC constitutes an open problem.

However, our main achievement is that the new construction does not require any DC structure of the function or its directional derivative. What is only needed to calculate the directed subdifferential are (recursively) the directional derivatives of the function and of their restrictions to lower-dimensional affine sets. The directed subdifferential can be constructed for any n times uniformly directionally differentiable function, which we call ‘directed subdifferentiable’. The directed subdifferential is defined inductively in the dimension n , analogously to the inductive definition of directed sets.

Let us note that other widely used subdifferentials can be evaluated via the directed subdifferential: Dini/Fréchet, Michel-Penot, and, in some cases also, Clarke and Mordukhovich subdifferentials (see [4]). This together with the exact calculus rules for the directed subdifferential may help in practical applications where computing nonconvex subdifferentials is needed.

The paper is organized as follows: in Section 2 we remind the basic ideas behind the Banach space of the directed sets and in Section 3 we briefly introduce the directed subdifferential, if a DC structure for the function or its directional derivative is known. To give the reader the flavour of our

results, and to explain our motivation, in Section 4 we demonstrate how to construct the directed subdifferential for DC functions without knowing the DC decomposition. Section 5 is devoted to ‘directed subdifferentiable’ functions and a rigorous definition of the directed subdifferential for such functions is given there, as well as calculus rules and one example of a directed differentiable function which is not quasidifferentiable.

2 Directed sets – a brief overview

Here we only sketch the most important notions related to directed subdifferential. For more details and examples we refer the reader to [1, 2, 3, 5, 6].

We use the standard notation in the paper: by $\|\cdot\|$ we denote the Euclidean norm, and S_{n-1} is the $(n-1)$ -dimensional sphere in \mathbb{R}^n .

The directed sets and the *embedding* J_n of convex compact sets in \mathbb{R}^n into the Banach space of directed sets are introduced in [1, 2] recursively in the dimension n . The definition of the directed set is motivated by the Minkowski duality between compact convex sets and their support functions which correspond to the first and the second component in the definition. Let $C \subset \mathbb{R}^n$ be nonempty, convex and compact and let $l \in \mathbb{R}^n$. The *support function* and respectively the *supporting face* of C in direction l are defined by

$$\begin{aligned}\delta^*(l, C) &= \max_{c \in C} \langle l, c \rangle, \\ Y(l, C) &= \{y \in C \mid \langle l, y \rangle = \delta^*(l, C)\} = \arg \max_{c \in C} \langle l, c \rangle.\end{aligned}$$

Note that $Y(\cdot, C)$ is upper semicontinuous and for $l = 0$, $Y(l, C) = C$. By $y(l, C)$ we denote any point of the set $Y(l, C)$, and if the latter is a singleton (i.e. there is a unique supporting point), then $Y(l, C) = \{y(l, C)\}$.

In one dimension, the *directed embedded intervals* are defined by the values of the support function in the two unit directions ± 1 . Given an interval $[a, b] \in \mathbb{R}$, its embedded image in the space of one dimensional directed space is

$$\overrightarrow{[a, b]} = J_1([a, b]) = (\delta^*(\eta, [a, b]))_{\eta=\pm 1} = (-a, b) \quad (a \leq b).$$

A general *directed interval* $\overrightarrow{A}_1 = \overrightarrow{[c, d]} = (-c, d)$ allows that c, d are arbitrary real numbers, even $c > d$ is possible (see references in [1, 2]).

Linear operations on directed intervals are defined in a natural way and are motivated by the operations for vectors in \mathbb{R}^2 : for directed intervals $\overrightarrow{[a, b]}$ and $\overrightarrow{[c, d]}$ and any $\alpha, \beta \in \mathbb{R}$

$$\alpha \overrightarrow{[a, b]} + \beta \overrightarrow{[c, d]} = \overrightarrow{[\alpha a + \beta c, \alpha b + \beta d]}. \quad (1)$$

Definition 2.1. We call \overrightarrow{A} a directed set

- (i) in \mathbb{R} , if it is a directed interval. Its norm is $\|\overrightarrow{A}\|_1 = \max_{l=\pm 1} |a_1(l)|$.
- (ii) in \mathbb{R}^n , $n \geq 2$, if there exist a continuous function $a_n : S_{n-1} \rightarrow \mathbb{R}$ and a uniformly bounded map having lower-dimensional directed sets as images $\overrightarrow{A_{n-1}} : S_{n-1} \rightarrow \mathcal{D}(\mathbb{R}^{n-1})$ with respect to $\|\cdot\|_{n-1}$.

We denote $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l \in S_{n-1}}$ and define its norm as

$$\|\overrightarrow{A}\| := \|\overrightarrow{A}\|_n := \max\left\{ \sup_{l \in S_{n-1}} \|\overrightarrow{A_{n-1}(l)}\|_{n-1}, \max_{l \in S_{n-1}} |a_n(l)| \right\}.$$

The set of all directed sets in \mathbb{R}^n is denoted by $\mathcal{D}(\mathbb{R}^n)$.

Observe that the space of directed sets is a Banach space (see [1, Theorem 4.15]). The *linear operations* are defined recursively, on the two components of the directed sets $\overrightarrow{A} = (\overrightarrow{A_{n-1}(l)}, a_n(l))_{l \in S_{n-1}}$, $\overrightarrow{B} = (\overrightarrow{B_{n-1}(l)}, b_n(l))_{l \in S_{n-1}}$:

$$\boxed{\begin{aligned} \overrightarrow{A} + \overrightarrow{B} &:= (\overrightarrow{A_{n-1}(l)} + \overrightarrow{B_{n-1}(l)}, a_n(l) + b_n(l))_{l \in S_{n-1}}, \\ \lambda \cdot \overrightarrow{A} &:= (\lambda \cdot \overrightarrow{A_{n-1}(l)}, \lambda \cdot a_n(l))_{l \in S_{n-1}} \quad (\lambda \in \mathbb{R}), \\ \overrightarrow{A} - \overrightarrow{B} &:= \overrightarrow{A} + (-\overrightarrow{B}) = (\overrightarrow{A_{n-1}(l)} - \overrightarrow{B_{n-1}(l)}, a_n(l) - b_n(l))_{l \in S_{n-1}}. \end{aligned}} \quad (2)$$

The space of directed sets is hence a normed vector space with zero $\overrightarrow{0} = \overrightarrow{0_{\mathcal{D}(\mathbb{R}^n)}} = (0_{\mathcal{D}(\mathbb{R}^n)}, 0)_{l \in S_{n-1}}$ (see [1, Proposition 4.14]).

The *embedding* $J_n : \mathcal{C}(\mathbb{R}^n) \rightarrow \mathcal{D}(\mathbb{R}^n)$ of the convex sets in the space of directed sets which determines for every set $A \in \mathcal{C}(\mathbb{R}^n)$ its embedded image $\overrightarrow{A} \in \mathcal{D}(\mathbb{R}^n)$ is defined as follows:

- (i) For $n = 1$, $\overrightarrow{[a, b]} = J_1([a, b]) := (-a, b)$.

- (ii) For $n \geq 2$, $\vec{A} = J_n(A) := (J_{n-1}(\Pi_{n-1,l}(Y(l, A))), \delta^*(l, A))_{l \in S_{n-1}}$, where $J_{n-1}(\Pi_{n-1,l}(Y(l, A)))$ is an $n - 1$ -dimensional image of the face $Y(l, A)$, $\Pi_{n-1,l}(x) := \pi_{n-1,n} R_{n,l}(x)$ and $\pi_{n-1,n} \in \mathbb{R}^{(n-1) \times n}$ is the natural projection, i.e. $\pi_{n-1,n}(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1})$, and $R_{n,l}$ is a fixed rotation for every $l \in S_{n-1}$ satisfying

$$R_{n,l}(l) = e^n, \quad R_{n,l}(\text{span}\{l\}^\perp) = \text{span}\{e^1, e^2, \dots, e^{n-1}\}. \quad (3)$$

We also note that the embedding is positively linear by [1, Proposition 3.10 and Theorem 4.17], i.e.

$$J_n(\lambda A + \mu B) = \lambda J_n(A) + \mu J_n(B) \quad (A, B \in \mathcal{C}(\mathbb{R}^n), \lambda, \mu \geq 0). \quad (4)$$

3 The directed subdifferential with known DC structure

Before introducing the directed subdifferential, let us recall that the classical Moreau-Rockafellar *convex subdifferential* of a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at $x \in \mathbb{R}^n$ is a compact convex set

$$\partial f(x) := \{s \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n : \langle s, y - x \rangle + f(x) \leq f(y)\}. \quad (5)$$

In the sequel, the Moreau-Rockafellar *convex subdifferential of a sublinear function* g at zero is denoted by ∂g instead of $\partial g(0)$.

Note that $Y(l, C)$ equals the subdifferential of the support function of C at l [17, Corollary 23.5.3].

It is well-known (see e.g. [14, Chap. VI, Definition 1.1.4 and Example 3.1, (3.1)]) that

$$\delta^*(l, \partial f(x)) = f'(x; l), \quad (6)$$

$$\partial \delta^*(\cdot, A)(l) = Y(l, A) \quad (A \in \mathcal{C}(\mathbb{R}^n)), \quad (7)$$

and hence,

$$\partial f'(x; \cdot)(l) = Y(l, \partial f(x)). \quad (8)$$

where $f'(x; l)$ is the (Dini) *directional derivative* of f at x in a direction l ,

$$f'(x; l) = \lim_{t \rightarrow 0^+} \frac{f(x + tl) - f(x)}{t}.$$

The following statement is [12, Proposition 3.1 from Chap. I.3] and is needed later for proving the closure of arithmetic operations for directed subdifferentiable functions:

Remark 3.1. *Let $f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be directionally differentiable at $x \in \mathbb{R}^n$. Then their sum, product and quotient (if $f_2(x) \neq 0$) are also directionally differentiable at this point and the following formulas hold for $\alpha, \beta \in \mathbb{R}$:*

$$\begin{aligned} (\alpha f_1 + \beta f_2)'(x; l) &= \alpha f_1'(x; l) + \beta f_2'(x; l) , \\ (f_1 \cdot f_2)'(x; l) &= f_1(x) f_2'(x; l) + f_2(x) f_1'(x; l) , \\ \left(\frac{f_1}{f_2} \right)'(x; l) &= - \frac{f_1(x) f_2'(x; l) - f_2(x) f_1'(x; l)}{[f_2(x)]^2} \end{aligned}$$

A similar closedness property with respect to min/max operations is given in [12, Corollary 3.2 in Sect. I.3]:

Remark 3.2. *Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I = \{1, 2, \dots, p\}$ have finite Dini directional derivatives at $x \in \mathbb{R}^n$ for all $l \in \mathcal{S}_{n-1}$. Then the pointwise minimum and pointwise maximum functions $f_{\max}(\cdot) = \max_{i \in I} f_i(\cdot)$ and $f_{\min}(\cdot) = \min_{i \in I} f_i(\cdot)$ are also Dini directionally differentiable at x with*

$$f'_{\max}(x; l) = \max_{i \in I(x)} f'_i(x; l); \quad f'_{\min}(x; l) = \min_{i \in J(x)} f'_i(x; l) ,$$

where $I(x) = \{i \in I \mid f_i(x) = f_{\max}(x)\}$, $J(x) = \{i \in I \mid f_i(x) = f_{\min}(x)\}$.

Clearly, we can embed the Moreau-Rockafellar subdifferential in the space of directed sets, i.e.

$$\vec{\partial} f(x) = \begin{cases} (\delta^*(l, \partial f(x)))_{l=\pm 1} & \text{if } n = 1, \\ (J_{n-1}(\Pi_{n-1, l}(Y(l, \partial f(x)))), \delta^*(l, \partial f(x)))_{l \in \mathcal{S}_{n-1}} & \text{if } n \geq 2. \end{cases} \quad (9)$$

Moreover, using the embedding, we can define the directed subdifferential of a DC function [3].

Definition 3.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be DC, i.e. $f = g - h$, where $g, h : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex. The directed subdifferential of f at x is defined by*

$$\vec{\partial}_{DC} f(x) = J_n(\partial g(x)) - J_n(\partial h(x)) . \quad (10)$$

The directed subdifferential was extended to the larger class of quasidifferentiable functions in [5, 6]. The core idea of this generalization is that instead of DC functions more general functions are considered, whose directional derivatives are DC (see e.g. [12, Chap. 3] and [16, Sect. 10.2]). The directed subdifferential is then defined via the directional derivative.

Definition 3.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be quasidifferentiable (QD), i.e.*

$$f'(x; l) = \delta^*(l, \underline{\partial}f(x)) - \delta^*(l, \ominus \bar{\partial}f(x)) \quad (l \in S_{n-1})$$

with two convex, compact, nonempty sets $\underline{\partial}f(x), \bar{\partial}f(x) \subset \mathbb{R}^n$. The directed subdifferential of f at x is defined by

$$\vec{\partial}_{QD}f(x) = J_n(\underline{\partial}f(x)) - J_n(\ominus \bar{\partial}f(x)) . \quad (11)$$

Since

$$\underline{\partial}f(x) = \partial g(x) , \quad \bar{\partial}f(x) = \ominus \partial h(x)$$

for a DC function $f = g - h$, Definition 3.4 is an extension of the directed subdifferential for DC functions introduced in Definition 3.3, namely it holds for DC functions

$$\vec{\partial}_{QD}f(x) = \vec{\partial}_{DC}f(x) = \vec{\partial}f(x) . \quad (12)$$

We would like to turn our attention to quasidifferentiable functions and state the following lemma which is a slight extension of [4, Proposition 3.15] from convex to quasidifferentiable functions.

Lemma 3.5. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be quasidifferentiable. Then the directed subdifferential of f fulfills*

$$\vec{\partial}_{QD}f(x) = \vec{\partial}_{DC}f'(x; \cdot)(0) . \quad (13)$$

Proof. Setting $f'(x; l) = \tilde{g}(l) - \tilde{h}(l)$ with

$$\tilde{g}(l) = \delta^*(l, \underline{\partial}f(x)) , \quad \tilde{h}(l) = \delta^*(l, \ominus \bar{\partial}f(x)) ,$$

we can use (7) which yields

$$\partial \tilde{g}(l) = Y(l, \underline{\partial}f(x)) , \quad \partial \tilde{h}(l) = Y(l, \ominus \bar{\partial}f(x)) .$$

Proposition 4.3 applies and we can prove that

$$\begin{aligned}
\vec{\partial}_{DC} f'(x; \cdot)(0) &= J_n(\partial \tilde{g}(x)) - J_n(\partial \tilde{h}(x)) \\
&= J_n(Y(0, \underline{\partial} f(x))) - J_n(Y(0, \ominus \bar{\partial} f(x))) \\
&= J_n(\underline{\partial} f(x)) - J_n(\ominus \bar{\partial} f(x)) = \vec{\partial}_{QD} f(x) .
\end{aligned}$$

□

The previous result allows to carry over results on the directed subdifferential from the DC to the quasidifferentiable functions.

4 The directed subdifferential without a DC structure

In this section we suggest a technique for constructing the directed subdifferential only from the directional derivative, i.e. without using any information on the DC structure of the function and avoiding the direct embedding of the relevant subdifferentials. For the sake of clarity, we start with convex functions, then extend the construction to DC functions, and end up with defining a more general class of ‘directed subdifferentiable’ functions.

Starting from the equality (9) of the embedded subdifferential of convex analysis, we now obtain an explicit representation of the directed subdifferential of a convex function via its directional derivatives. Here, we can apply (6) to see that the second component of the directed subdifferential in (9) equals the directional derivative. To obtain an explicit formula for the first component as well, we use an auxiliary function $f_l(\cdot)$ which depends only on $n - 1$ variables.

We first gather the result for convex functions and apply the new representation to DC and quasidifferentiable functions in a second step.

4.1 Convex functions

Recall that the directed subdifferential of a convex function is the embedded image of its Moreau-Rockafellar subdifferential into the space of directed sets. We describe its construction first in one dimension.

It is not difficult to observe that for any convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $[a, b] := \partial f(x)$ we have

$$f'(x; -1) = \max_{v \in [a, b]} \{-v\} = -a; \quad f'(x; 1) = \max_{v \in [a, b]} \{v\} = b,$$

and hence, by the definition of the directed subdifferential,

$$\vec{\partial}_{DC} f(x) = J_1(\partial f(x)) = \overrightarrow{[a, b]} = \overrightarrow{[-f'(x; -1), f'(x; 1)]}. \quad (14)$$

Based on this construction, we now obtain a similar representation for higher dimensions. Consider a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and fix a point $x \in \mathbb{R}^n$. Let $f'(x; \cdot)$ be the directional derivative of f at x . To define an explicit expression of the first term in the right-hand side of (9) for a given vector $l \in S_{n-1}$ we define an auxiliary function $f_l : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ which is a restriction of $f'(x; \cdot)$ to the hyperplane $l + \text{span}\{l\}^\perp$, i.e. for $y \in \mathbb{R}^{n-1}$ let

$$f_l(y) = f'(x; l + \Pi_{n-1, l}^T y), \quad (15)$$

where $\Pi_{n-1, l}^T = R_{n, l}^{-1} \pi_{n-1, n}^T$ maps \mathbb{R}^{n-1} to $\text{span}\{l\}^\perp$. Here $\pi_{n-1, n}^T$ maps points in \mathbb{R}^{n-1} to \mathbb{R}^n by adding a zero last coordinate, and $R_{n, l}^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the inverse of the fixed rotation introduced in Section 2 (see (3)). Thus,

$$R_{n, l}^{-1}(e^n) = l, \quad R_{n, l}^{-1}(\text{span}\{e_1, e_2, \dots, e_{n-1}\}) = \text{span}\{l\}^\perp.$$

In the following proposition the first (lower-dimensional) component of the directed subdifferential is expressed via a directional derivative of a certain function restricted to an affine space.

Proposition 4.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex. Then,*

$$\vec{\partial}_{DC} f(x) = \begin{cases} (f'(x; l))_{l=\pm 1} & \text{for } n = 1, \\ \left(\vec{\partial}_{DC} f_l(0), f'(x; l) \right)_{l \in S_{n-1}} & \text{for } n \geq 2. \end{cases} \quad (16)$$

Proof. Taking (14) into account, we only prove the case $n \geq 2$.

First observe that the directional derivative $f'(x; l)$ of a convex function is also convex as a function of the direction l , therefore $f'(x; l)$ is directionally differentiable, and the higher directional derivatives as well. Hence, in the case of convex functions directional derivatives of any ‘order’ are defined.

Since f is convex, we can apply (6)–(8) and it follows that

$$\begin{aligned} f'_l(0; u) &= \lim_{t \rightarrow 0^+} \frac{f_l(tu) - f_l(0)}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{f'(x; l + t\Pi_{n-1,l}^T u) - f'(x; l)}{t} \\ &= [f'(x; \cdot)]'(l; \Pi_{n-1,l}^T u) . \end{aligned}$$

Using the equality $[f'(x; \cdot)]'(l; v) = \delta^*(v; Y(l, \partial f(x)))$ (see e.g. [5, Lemma 3.4]), we get

$$\begin{aligned} f'_l(0; u) &= \max_{\zeta \in Y(l, \partial f(x))} \zeta^T \Pi_{n-1,l}^T u \\ &= \max_{\zeta \in Y(l, \partial f(x))} (\Pi_{n-1,l} \zeta)^T u \\ &= \max_{\xi \in \Pi_{n-1,l} Y(l, \partial f(x))} \xi^T u = \delta^*(u; \Pi_{n-1,l} Y(l, \partial f(x))) . \end{aligned}$$

From here, by (6) we get $\Pi_{n-1,l} Y(l, \partial f(x)) = \partial f_l(0)$, hence

$$\overrightarrow{\partial}_{DC} f_l(0) = J_{n-1}(\Pi_{n-1,l} Y(l, \partial f(x))) .$$

This, together with (6) and (9) implies (16). \square

We illustrate our construction with a simple example.

Example 4.2. Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x) = \max\{|x_1|, |x_2|\}$, and for any $l \in \mathcal{S}_1$ let

$$R_{2,l} = \begin{bmatrix} l_2 & -l_1 \\ l_1 & l_2 \end{bmatrix} , \text{ then } R_{2,l}^{-1} = \begin{bmatrix} l_2 & l_1 \\ -l_1 & l_2 \end{bmatrix} .$$

We first construct the directed subdifferential by embedding the subdifferential of f into the space of the directed sets as in the original definition of the directed subdifferential, and then show that the same result can be obtained via the directional derivative approach.

Since f is positively homogeneous, $f'(0; p) = f(p)$ for all $p \in \mathbb{R}^2$; besides it is not difficult to see (e.g. applying [14, Corollary 4.3.2 from Chap. VI]) that the grey polytope in Fig. 1(a) equals

$$\partial f = \partial f(0) = \text{co}\{(1, 0), (0, 1), (-1, 0), (0, -1)\} .$$

By equality (9) we have

$$\vec{\partial}_{DC}f(0) = J_2(\partial f) = (J_1(\Pi_{1,l}(Y(l, \partial f))), \delta^*(l; \partial f))_{l \in \mathcal{S}_1} . \quad (17)$$

It is not difficult to observe that for $l \in \mathcal{S}_1$

$$Y(l, \partial f) = \begin{cases} \{(1, 0)\} , & l_1 > |l_2| , \\ \{(-1, 0)\} , & l_1 < -|l_2| , \\ \{(0, 1)\} , & l_2 > |l_1| , \\ \{(0, -1)\} , & l_2 < -|l_1| , \\ \text{co}\{(1, 0), (0, 1)\} , & l = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) , \\ \text{co}\{(-1, 0), (0, 1)\} , & l = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) , \\ \text{co}\{(-1, 0), (0, -1)\} , & l = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) , \\ \text{co}\{(1, 0), (0, -1)\} , & l = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) . \end{cases}$$

Since

$$\Pi_{1,l} = \pi_{1,2}R_{2,l} = [1, 0] \begin{bmatrix} l_2 & -l_1 \\ l_1 & l_2 \end{bmatrix} = [l_2, -l_1] ,$$

we have for $l \in \mathcal{S}_1$

$$\vec{D}_l := J_1(\Pi_{1,l}(Y(l, \partial f))) = \begin{cases} \overrightarrow{\{l_2\}} , & l_1 > |l_2| , \\ \overrightarrow{\{-l_2\}} , & l_1 < -|l_2| , \\ \overrightarrow{\{-l_1\}} , & l_2 > |l_1| , \\ \overrightarrow{\{l_1\}} , & l_2 < -|l_1| , \\ \overrightarrow{\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]} , & |l_1| = |l_2| . \end{cases} \quad (18)$$

The construction of the directed subdifferential is illustrated on Fig. 1.

Noticing that by $f'(0; l) = \delta^*(l; \partial f)$, we have

$$\vec{\partial}_{DC}f(0) = \left(\vec{D}_l, f'(0; l) \right)_{l \in \mathcal{S}_1} \quad (19)$$

from (17) and (18), where \vec{D}_l is given by (18).

We now obtain the directed subdifferential via the directional derivative based procedure. Observe that for every $y \in \mathbb{R}$

$$l + \Pi_{1,l}^T y = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} + \begin{bmatrix} l_2 & l_1 \\ -l_1 & l_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} y = \begin{bmatrix} l_1 + yl_2 \\ l_2 - yl_1 \end{bmatrix} ,$$

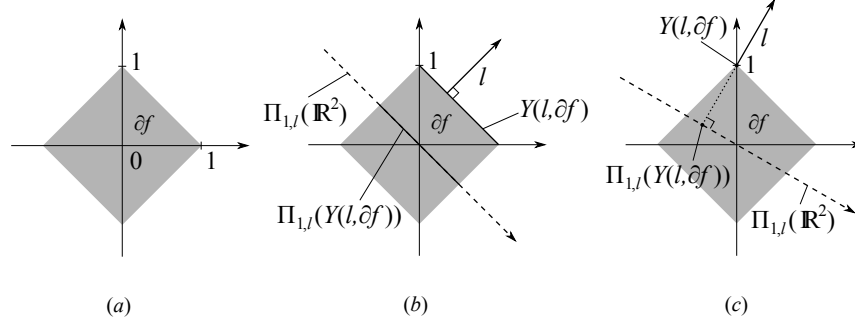


Figure 1: Construction of the directed subdifferential via embedding of ∂f (Example 4.2): a) the subdifferential of f at 0_2 ; b) projecting the supporting faces onto the one-dimensional subspace $\Pi_{1,l}(\mathbb{R}^2)$ for $l = (1/\sqrt{2}, 1/\sqrt{2})$ and c) for $l = (1/2, \sqrt{3}/2)$

hence,

$$f_l(y) = f'(0; l + \Pi_{1,l}^T y) = f(l + \Pi_{1,l}^T y) = \max\{|l_1 + y l_2|, |l_2 - y l_1|\} .$$

An illustration of how to construct f_l is given on Fig. 2.

It is a trivial exercise to compute

$$f'_l(0; 1) = \begin{cases} l_2, & l_1 > |l_2|, \\ -l_2, & l_1 < -|l_2|, \\ -l_1, & l_2 > |l_1|, \\ l_1, & l_2 < -|l_1|, \\ \frac{1}{\sqrt{2}}, & |l_1| = |l_2|, \end{cases} \quad f'_l(0; -1) = \begin{cases} -l_2, & l_1 > |l_2|, \\ l_2, & l_1 < -|l_2|, \\ l_1, & l_2 > |l_1|, \\ -l_1, & l_2 < -|l_1|, \\ \frac{1}{\sqrt{2}}, & |l_1| = |l_2|. \end{cases}$$

Applying (14) to compute $\vec{\partial}_{DC} f_l(0)$, we observe that $\vec{\partial}_{DC} f_l(0) = \vec{D}_l$, where \vec{D}_l is defined by (18). Therefore, by (16)

$$\vec{\partial}_{DC} f(x) = \left(\vec{\partial}_{DC} f_l(0), f'(x; l) \right)_{l \in \mathcal{S}_1} = \left(\vec{D}_l, f'(x; l) \right)_{l \in \mathcal{S}_1} ,$$

which coincides with (19).

In the next section we show that the directed subdifferential of a DC and a QD function can also be constructed via directional derivatives, i.e. Proposition 4.1 also holds for DC and QD functions.

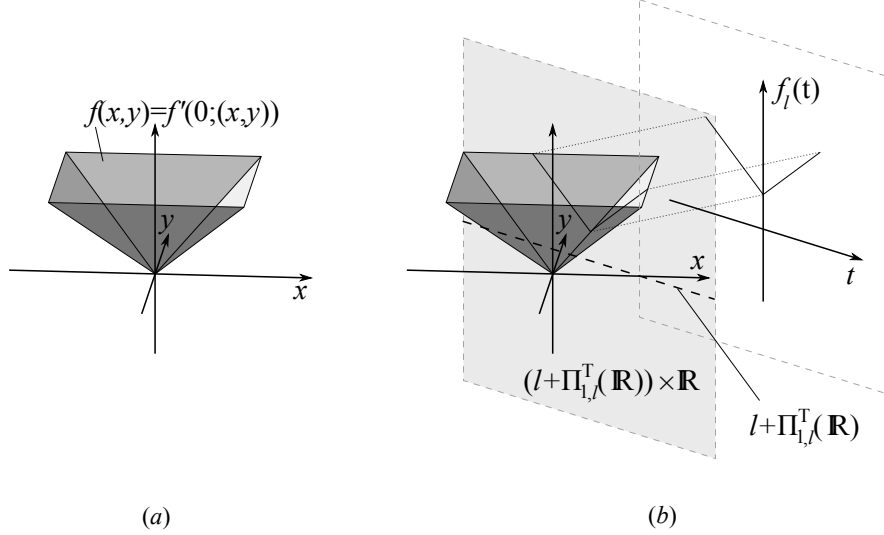


Figure 2: Construction of the restriction f_l of the function f (Example 4.2) for $l = (1/\sqrt{2}, 1/\sqrt{2})$: a) graph of the function f which in this case coincides with $f'(0_2; \cdot)$; b) the restriction of f to $l + \Pi_{1,l}^T(\mathbb{R})$

4.2 DC and quasidifferentiable functions

Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is DC with $f = g - h$ (where $g, h : \mathbb{R} \rightarrow \mathbb{R}$ are convex), then for any $x \in \mathbb{R}$ using (14) we get

$$\begin{aligned}
 \overrightarrow{\partial}_{DC} f(x) &= J_1(\partial g(x)) - J_1(\partial h(x)) \\
 &= \overrightarrow{[-g'(x; -1), g'(x; 1)]} - \overrightarrow{[-h'(x; -1), h'(x; 1)]} \\
 &= \overrightarrow{[-(g'(x; -1) - h'(x; -1)), g'(x; 1) - h'(x; 1)]} \\
 &= \overrightarrow{[-f'(x; -1), f'(x; 1)]}, \tag{20}
 \end{aligned}$$

since $f'(x; \cdot) = g'(x; \cdot) - h'(x; \cdot)$ (see e.g. Remark 3.1). Hence, $\overrightarrow{\partial}_{DC} f(x) = \overrightarrow{[-f'(x; -1), f'(x; 1)]}$, which coincides with the expression (14).

Now let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be DC, and fix a point $x \in \mathbb{R}^n$. Let $f'(x; \cdot)$ be the directional derivative of f at x . Consider the restriction f_l of the directional derivative defined by (15).

Proposition 4.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be DC with $f = g - h$ with convex*

functions $g, h : \mathbb{R}^n \rightarrow \mathbb{R}$. Then,

$$\vec{\partial}_{DC} f(x) = \begin{cases} (f'(x; l))_{l=\pm 1} & \text{for } n = 1, \\ \left(\vec{\partial}_{DC} f_l(0), f'(x; l) \right)_{l \in S_{n-1}} & \text{for } n \geq 2. \end{cases} \quad (21)$$

Proof. Since (20) is already proved, we only consider the case $n \geq 2$.

By Remark 3.1 $f'(x; l) = g'(x; l) - h'(x; l)$, and hence for any $y \in \mathbb{R}^{n-1}$

$$f_l(y) = g'(x; l + \Pi_{n-1, l}^T y) - h'(x; l + \Pi_{n-1, l}^T y) = g_l(y) - h_l(y) .$$

Since both g and h are convex, we can use the results from the previous section, i.e. $\partial g_l = \Pi_{n-1, l} Y(l; \partial g(x))$ and $\partial h_l = \Pi_{n-1, l} Y(l; \partial h(x))$. Observe that f_l is also DC, since g_l and h_l are convex and $f_l = g_l - h_l$. Therefore, the directed subdifferential of f_l at 0 is the difference of the convex embeddings of ∂g_l and ∂h_l :

$$\begin{aligned} \vec{\partial}_{DC} f_l &= J_{n-1}(\partial g_l) - J_{n-1}(\partial h_l) \\ &= J_{n-1}(\Pi_{n-1, l} Y(l; \partial g(x))) - J_{n-1}(\Pi_{n-1, l} Y(l; \partial h(x))) ; \end{aligned} \quad (22)$$

moreover,

$$\delta^*(l; \partial g(x)) - \delta^*(l; \partial h(x)) = g'(x; l) - h'(x; l) = f'(x; l) . \quad (23)$$

Now comparing (22) and (23) with the definition of the directed subdifferential, we obtain (21). \square

The case of quasidifferentiable functions is a consequence of the DC case by taking into account Lemma 3.5:

Corollary 4.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be quasidifferentiable with*

$$f'(x; l) = \delta^*(l, \underline{\partial} f(x)) - \delta^*(l, \overline{\partial} f(x)) \quad (l \in S_{n-1})$$

and $\underline{\partial} f(x), \overline{\partial} f(x) \in \mathcal{C}(\mathbb{R}^n)$. Then,

$$\vec{\partial}_{QD} f(x) = \begin{cases} (f'(x; l))_{l=\pm 1} & \text{for } n = 1, \\ \left(\vec{\partial}_{QD} f_l(0), f'(x; l) \right)_{l \in S_{n-1}} & \text{for } n \geq 2. \end{cases} \quad (24)$$

Proof. Let $\varphi(l) = f'(x; l)$, then $\varphi(\cdot)$ is a DC function. It is easy to conclude that

$$\begin{aligned}\varphi(0) &= f'(x; 0) = 0, \\ \varphi(h\eta) &= f'(x; h\eta) = hf'(x; \eta) = h\varphi(\eta), \\ \varphi'(0; \eta) &= \lim_{h \downarrow 0} \frac{1}{h}(\varphi(h\eta) - \varphi(0)) = \varphi(\eta) = f'(x; \eta), \\ \varphi_l(0) &= \varphi'(0; l + \Pi_{n-1, l}^T \eta) = f'(x; l + \Pi_{n-1, l}^T \eta) = f_l(0).\end{aligned}$$

Proposition 4.3 and Lemma 3.5 apply which show

$$\begin{aligned}\vec{\partial}_{DC}\varphi(0) &= \begin{cases} (\varphi'(0; l))_{l=\pm 1} & \text{for } n = 1, \\ \left(\vec{\partial}_{DC}\varphi_l(0), \varphi'(0; l)\right)_{l \in S_{n-1}} & \text{for } n \geq 2 \end{cases} \\ &= \begin{cases} (f'(x; l))_{l=\pm 1} & \text{for } n = 1, \\ \left(\vec{\partial}_{DC}f_l(0), f'(x; l)\right)_{l \in S_{n-1}} & \text{for } n \geq 2 \end{cases} \\ &= \vec{\partial}_{QD}f(x).\end{aligned}$$

□

The previous Proposition 4.3 and Corollary 4.4 can be seen as generalizations of [4, Lemma 3.10 and Theorem 3.11] to the case of difference of two sublinear functions (DS) functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ for state dimensions $n \geq 2$ and to DC/quasidifferentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ for $n \in \mathbb{N}$ (in the additional view of [4, Lemmas 3.6, 3.8 and 3.9]). In [4, Theorem 3.11] the directed subdifferential for such functions is expressed with the limes superior of Fréchet subdifferentials as first component and the directional derivative as second component (see [4] for more details). But in the latter article, the primal aim was to discover the connection of the Mordukhovich/basic subdifferentials to the Rubinov subdifferential (the visualization of the directed subdifferential).

Hence, we can give an equivalent definition of the directed subdifferential via directional derivatives for both cases (DC and quasidifferentiable functions). Moreover, it appears that in contrast to the original definitions of the directed subdifferential in [3, 5], these results lead to a new definition of the directed subdifferential for an extended class of functions wider than DC or QD functions. We first need to define such functions.

5 Extension of the directed subdifferential

In this section, we give a definition of ‘directed subdifferentiable’ functions and show that a directed subdifferential can be defined for such functions even though they are not necessarily DC (and even not quasidifferentiable).

5.1 Directed subdifferentiable functions

We define the ‘directed subdifferentiable’ functions recursively by considering restrictions of directional derivatives to affine subspaces in a recursive way.

Definition 5.1. (directed subdifferentiable function) *Given a constant $M \geq 0$, a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called M -directed subdifferentiable at $x \in \mathbb{R}$ if both its left- and right-sided derivatives exist and*

$$\max\{|f'(x; -1)|, |f'(x; 1)|\} \leq M .$$

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $n \geq 2$ is called M -directed subdifferentiable at $x \in \mathbb{R}^n$ if its directional derivative $f'(x; l)$ exists for every l , is continuous as a function of the direction l , is bounded by M :

$$\max_{l \in S_{n-1}} |f'(x; l)| \leq M ,$$

and the restriction $f_l(\cdot) = f'(x; l + \Pi_{n-1, l}^T(\cdot)) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ of its directional derivative to $l + \text{span}\{l\}^\perp$ is also M -directed subdifferentiable at 0_{n-1} for all $l \in S_{n-1}$.

We say that a function is directed subdifferentiable if it is M -directed subdifferentiable for some $M \geq 0$.

Observe that the restrictions f_l are constructed in the same way as for convex and DC functions above (also see Fig. 2 for an illustration).

Remark 5.2. *Even though the above definition might look cumbersome, the directed subdifferentiable functions are very common in applications. All Lipschitz functions definable on o-minimal structure (which include semi-algebraic functions [11, 15]) are directed subdifferentiable. Even though this fact is fairly obvious for anybody sufficiently familiar with definable functions, we provide a sketch of the proof for convenience in Lemma 5.4. We would like to thank Aris Daniilidis and Dmitry Drusviatskiy for providing us with helpful hints for the proof.*

Remark 5.3. *We would like to mention that the study of definable functions and their generalized subdifferentials has recently drawn much attention (e.g. see [8, 11, 13]). The class of definable functions is very broad, and vast majority of functions encountered in applications belongs to this class. This restriction helps focussing on results crucial for optimization theory [15] without being distracted by abnormal behaviour of exotic examples.*

The fact that locally Lipschitz definable functions are directed subdifferentiable follows from the next lemma. One needs to apply it recursively n times to the directional derivatives of directional derivatives to obtain the desired result. We only provide a sketch of the proof, as it is not the core result of the paper, and, as we mentioned before, the proof is evident for anybody familiar with o-minimal structures [10, 15].

Lemma 5.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz around $x \in \mathbb{R}^n$ and definable on some o-minimal structure. Then the directional derivative $f'(x; \cdot)$ of f at x exists, is globally Lipschitz and definable on the whole space \mathbb{R}^n .*

Proof. The existence of the directional derivative follows from the monotonicity theorem (see [7, Chap. 5] and [10, Chap. 2]). It is a trivial exercise to show that the function $g_x(y, t) : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$,

$$g_x(y, t) = \frac{f(x + ty) - f(x)}{t}$$

is definable (it is a composition of definable functions). The monotonicity theorem implies that for every $y \in \mathbb{R}^n$ there exists $r_1 > 0$ such that g_x is monotone on $y \times (0, r_1)$. Moreover, since f is locally Lipschitz around x , there is an $r_2 > 0$ such that g is bounded on $\{y\} \times (0, r_2)$. Hence the limit

$$f'(x; y) = \lim_{t \downarrow 0} g_x(y, t)$$

exists for every $y \in \mathbb{R}^n$.

The Lipschitzness of the directional derivative follows from the local Lipschitzness of the function f around x .

It remains to show that the directional derivative $h_x : \mathbb{R}^n \rightarrow \mathbb{R}$, $h_x(y) = f'(x; y)$ is definable. We use the argument from [15]. For any definable $\psi : \mathbb{R}^n \times A \rightarrow \mathbb{R}$, where $A \in \mathbb{R}^p$ is a definable set, the inf function

$$\varphi(x) = \inf_{s \in A} \psi(x, s)$$

is definable (this result is not difficult to prove using the standard technique described, e.g. in [10, proof of Proposition 1.2]). Observe that

$$\begin{aligned}
h_x(y) &= f(x; y) = \lim_{t \downarrow 0} g_x(y, t) \\
&= \sup_{\varepsilon \in (0, r_1)} \inf_{t \in (0, \varepsilon)} g_x(y, t) \\
&= \sup_{\varepsilon \in (0, r_1)} \inf_{t \in (0, +\infty)} g_x(y, \min\{t, \varepsilon\}) \\
&= - \inf_{\varepsilon \in (0, r_1)} \left[- \inf_{t \in (0, +\infty)} g_x(y, \min\{t, \varepsilon\}) \right].
\end{aligned}$$

The function $g_x(y, \min\{t, \varepsilon\}) : \mathbb{R}^n \times (0, +\infty) \times (0, +\infty)$ is definable, hence, $h_x(y)$ is definable as an inf function. \square

We next show that arithmetic operations on directed subdifferentiable functions as well as pointwise min and max operations again yield directed subdifferentiable functions.

Lemma 5.5. *Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be M_i -directed subdifferentiable at x for $i = 1, 2$ and $\alpha, \beta \in \mathbb{R}$. Then*

(i) $\alpha f_1 + \beta f_2$ is $|\alpha|M_1 + |\beta|M_2$ -directed subdifferentiable at x ,

(ii) $f_1 \cdot f_2$ is M -directed subdifferentiable at x with

$$M = M_1|f_2(x)| + M_2|f_1(x)|.$$

(iii) f_1/f_2 is M -directed subdifferentiable at x with

$$M = \frac{M_1|f_2(x)| + M_2|f_1(x)|}{|f_2(x)|^2},$$

if $f_2(x) \neq 0$.

Proof. Let $f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be M_1 and M_2 -directed subdifferentiable respectively, $x \in \mathbb{R}^n$ and let $f = \alpha f_1 + \beta f_2$. Then from Remark 3.1

$$f'(x; l) = \alpha f'_1(x; l) + \beta f'_2(x; l) \quad \forall l \in \mathbb{R}^n, \quad (25)$$

and therefore

$$\begin{aligned}
\max_{l \in S_{n-1}} |f'(x; l)| &\leq |\alpha| \max_{l \in S_{n-1}} |f'_1(x; l)| + |\beta| \max_{l \in S_{n-1}} |f'_2(x; l)| \\
&\leq M := |\alpha|M_1 + |\beta|M_2.
\end{aligned}$$

Moreover, (25) yields

$$f_l(\cdot) = \alpha(f_1)_l(\cdot) + \beta(f_2)_l(\cdot) .$$

Since both $(f_1)_l(\cdot)$ and $(f_2)_l(\cdot)$ are directionally differentiable, by the same argument as above we have

$$\max_{y \in S_{n-2}} |f'_l(0; y)| \leq M = |\alpha|M_1 + |\beta|M_2 \quad \forall l \in S_{n-1} .$$

Repeating the same argument over and over again until we reach the relevant one-dimensional restrictions, we prove that f is M -directed subdifferentiable.

The proof for the rest of the cases is analogous, except for several details that we highlight below.

Observe that for the product $f = f_1 \cdot f_2$ of two directed subdifferentiable functions we have from Remark 3.1

$$\begin{aligned} \sup_{l \in S_{n-1}} |f'(x; l)| &\leq |f_1(x)| \sup_{l \in S_{n-1}} |f'_2(x; l)| + |f_2(x)| \sup_{l \in S_{n-1}} |f'_1(x; l)| \\ &\leq |f_1(x)|M_1 + |f_2(x)|M_2 ; \end{aligned}$$

as we go one level down we get for each $l \in S_{n-1}$

$$\begin{aligned} \sup_{y \in S_{n-2}} |f'_l(0; y)| &= \sup_{y \in S_{n-2}} |f_1(x)(f_1)'_l(0; y) + f_1(x)(f_2)'_l(0; y)| \\ &\leq |f_1(x)|M_1 + |f_2(x)|M_2 ; \end{aligned}$$

and so on until dimension 1. The derivation is analogous for the ratio. \square

It can be shown with Remark 3.2 that the pointwise maximum and minimum functions are directed subdifferentiable.

Lemma 5.6. *Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ be M_i -directed subdifferentiable at x for $i \in I$, where I is a finite index set. Then the pointwise maximum function*

$$f_{\max}(x) = \max_{i \in 1:p} f_i(x)$$

is M_{\max} -directed subdifferentiable at x with

$$M_{\max} = \max_{i \in I(x)} |M_i|, \text{ with } I(x) = \{i \in I \mid f_i(x) = f_{\max}(x)\} ,$$

and likewise the pointwise minimum function

$$f_{\min}(x) = \min_{i \in 1:p} f_i(x)$$

is M_{\min} -directed subdifferentiable with

$$M_{\min} = \max_{i \in J(x)} |M_i|, \text{ with } J(x) = \{i \in J \mid f_i(x) = f_{\min}(x)\}.$$

We omit the proof of this Lemma, as it is analogous to the proof of Lemma 5.5, using the previous Remark 3.2.

5.2 Extension to directed subdifferentiable functions

The directed subdifferential is now extended to the class of directed subdifferentiable functions in which no DC structure for the function or the directional derivative is present any longer. Instead, we will base the construction on directional derivatives only.

Definition 5.7. (Second definition of the directed subdifferential via directional derivatives) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be directed subdifferentiable, then for any fixed $x \in \mathbb{R}^n$ the directed subdifferential of f at x can be defined as follows:

(i) For $n = 1$ let $\vec{\partial} f(x) = \overline{[-f'(x; -1); f'(x; 1)]}$;

(ii) For $n \geq 2$ let

$$\vec{\partial} f(x) = \left(\vec{\partial} f_l, f'(x; l) \right)_{l \in S_{n-1}},$$

where $f_l : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is defined by $f_l(\cdot) = f'(x; l + \Pi_{n-1, l}^T(\cdot))$.

The next statement shows that the directed subdifferential is a well-defined directed set for directed subdifferentiable functions.

Lemma 5.8. Let $x \in \mathbb{R}^n$ and assume $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is directed subdifferentiable. Then the object described by Definition 5.7 is a directed set. Besides, if f is M -directed subdifferentiable, then

$$\|\vec{\partial} f(x)\| = \inf\{M \mid f \text{ is } M\text{-directed subdifferentiable at } x\}, \quad (26)$$

moreover, the infimum is attained.

Proof. For $n = 1$ the statement is trivial: indeed, given $f : \mathbb{R} \rightarrow \mathbb{R}$ such that f is M -directed subdifferentiable at $x \in \mathbb{R}$, we have

$$\vec{\partial} f(x) = \overrightarrow{[-f'(x; -1); f'(x; 1)]}.$$

Clearly, $\|\vec{\partial} f(x)\| = \max\{|f'(x; -1)|, |f'(x; 1)|\} \leq M$.

Assume now that the statement is true for $n = k - 1$. We will prove it for $n = k$. Consider $f : \mathbb{R}^k \rightarrow \mathbb{R}$ such that f is M -directed subdifferentiable at $x \in \mathbb{R}^k$. By Definition 5.7 of the directed subdifferential

$$\vec{\partial} f(x) = \left(\vec{\partial} f_l, f'(x; l) \right)_{l \in S_{k-1}}.$$

Observe that for every $l \in S_{k-1}$ the function f_l is M -directed subdifferentiable, and hence by the induction assumption $\vec{\partial} f_l$ is a directed set, moreover,

$$\|\vec{\partial} f_l\| \leq M \quad \forall l \in S_{k-1}, \quad (27)$$

and hence the first component of $\vec{\partial} f_l$ in (26) is a bounded map from S_{k-1} to $\mathcal{D}(\mathbb{R}^{k-1})$. Further, since f is M -directed subdifferentiable, its directional derivative $f'(x; \cdot)$ is continuous and bounded by M on the unit sphere:

$$\sup_{l \in S_{k-1}} |f'(x; l)| \leq M. \quad (28)$$

From (27) and (28) we have $\|\vec{\partial} f(x)\| \leq M$.

It is self-evident from the definition of an M -directed subdifferentiable function that the infimum in (26) is attained. \square

We have already shown in Corollary 4.4 that Definition 5.7 extends the directed subdifferential for quasidifferentiable functions introduced in [5] and restated in Definition 3.4.

We would like to point out that the difficulty to construct the ‘max+min’ representation for the directional derivative has been repelling people from applying quasidifferentials to solve nonsmooth optimization problems (apart from the problem of non-uniqueness and ‘inflation’ after applying calculus rules already addressed in [6]). On the other hand, the excellent calculus rules of the quasidifferential, e.g. the exact sum rule, are often not taken into account. Therefore, the QD representation can be gained by applying the exact calculus rules. Assuming that the function consists of (nonsmooth) outer

operations like ‘max or min’ and of (differentiable) simple inner functions, one can iterate the QD representation formulas from the inner functions to end up with the QD representation of the (complicated) outer functions. This idea (similar to the ideas in algorithmic/automatic differentiation) can also be applied for the directed subdifferential. While the directed subdifferential is essentially based on the same idea as quasidifferential, our approach solves the fundamental problem of constructing the DC representation of the directional derivative — we simply do not need it, as we only utilize directional derivatives in our construction.

5.3 The class of directed subdifferentiable functions

We state in the following remark a list of functions classes that are directed subdifferentiable.

Remark 5.9. *The following types of functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ are directed subdifferentiable:*

- (i) *convex functions*
- (ii) *DC functions*
- (iii) *quasidifferentiable functions*
- (iv) *amenable (and strongly amenable) functions*
- (v) *lower- C^k and upper- C^k functions for $k \in \mathbb{N}$*

The first three classes belong to the class of directed subdifferentiable functions by [5, Lemma 3.4] which states that a directional derivative of a quasidifferentiable function is also quasidifferentiable.

(iv) follows from [5, Corollary 5.8], while (v) follows from [5, Corollary 5.4] for $k \geq 2$ and from [5, Proposition 5.5] for $k = 1$. In both cases it was shown that the function classes (iv)–(v), see the references in [5] with the definitions, belong to the bigger class of quasidifferentiable functions.

Curiously enough, it appears that the class of directed subdifferentiable functions is substantially wider than QD. We next give an example of a directed subdifferentiable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as

$$f(x, y) = \inf_{k \in 1:\infty} \left| x - \frac{y}{k} \right|$$

which is not quasidifferentiable. This particular example is motivated by an idea of J. C. H. Pang and based on the well-known one-dimensional construction discussed in detail in [5, Example 3.5] (see also the references therein).

The idea presented in this example can be seen as a prototype to define a family of functions that are not quasidifferentiable but directed subdifferentiable. In this sense it is a universal construction to create further examples.

To show that this function is Lipschitz, we need a small technical result which also states the Lipschitz constant. The proof is self-evident, hence, omitted; for a more general result with upper semi-continuous parameters see [9, Sec. 2.8].

Lemma 5.10. *Let $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, 2$, be globally Lipschitz with a common constant L . Then their pointwise maximum and minimum are also Lipschitz with the same constant.*

Example 5.11. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined as follows:*

$$f(x, y) = \inf_{k \in \mathbb{N}} \left| x - \frac{y}{k} \right|$$

(i) *Observe that for every $k \in \mathbb{N}$ the linear function $x - y/k$ is globally Lipschitz with the constant $L_k = \sqrt{1 + 1/k^2} \leq \sqrt{2} =: L$.*

From Lemma 5.10 we deduce that for any $N \in \mathbb{N}$ the function

$$f_N(x, y) = \min_{k \in \mathbb{N}} \left| x - \frac{y}{k} \right| = \min_{k \in \mathbb{N}} \max \left\{ x - \frac{y}{k}, -x + \frac{y}{k} \right\}$$

is Lipschitz with constant L .

Pick an arbitrary pair of points $x_1, x_2 \in \mathbb{R}^2$. It is not difficult to observe that

$$f(x, y) = \lim_{N \rightarrow \infty} f_N(x, y) ,$$

therefore, for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that

$$|f(x_1) - f_M(x_1)| < \frac{\varepsilon}{2} \|x_1 - x_2\|, \quad |f(x_2) - f_M(x_2)| < \frac{\varepsilon}{2} \|x_1 - x_2\| .$$

Therefore,

$$|f(x_1) - f(x_2)| \leq \varepsilon \|x_2 - x_1\| + |f_M(x_1) - f_M(x_2)| \leq (L + \varepsilon) \|x_2 - x_1\| ,$$

hence,

$$|f(x_1) - f(x_2)| \leq \lim_{\varepsilon \downarrow 0} (L + \varepsilon) \|x_2 - x_1\| = L \|x_2 - x_1\| ,$$

and by the arbitrariness of x_1 and x_2 the function f is globally Lipschitz with constant $L = \sqrt{2}$.

(ii) The function f is positively homogeneous, hence, it is Dini directionally differentiable at zero with

$$f'(0; l) = f(l) \quad \forall l \in \mathbb{R}^2.$$

It follows from the Lipschitzness of f that it is also Hadamard directionally differentiable.

(iii) We next show that even though this function is both Lipschitz and directionally differentiable, it is not quasidifferentiable. Assume the contrary: then there exist two sublinear functions $g, h : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$f'(0; (x, y)) = f(x, y) = g(x, y) - h(x, y) \quad \forall (x, y) \in \mathbb{R}^2.$$

Consider the restriction of this identity to the line $\mathbb{R} \times \{1\}$. Then

$$f(\cdot, 1) = \inf_{k \in \mathbb{N}} \left| \cdot - \frac{1}{k} \right| = g(\cdot, 1) - h(\cdot, 1),$$

where $g(\cdot, 1)$ and $h(\cdot, 1)$ are convex functions. It is well-known (see the discussion in [5] and references therein) that the function $f(\cdot, 1)$ is not of bounded variation, hence, can not be represented as a DC function. This contradicts our assumption that f is quasidifferentiable.

(iv) We next show that the function f is M -directed subdifferentiable at the origin with $M = \sqrt{2}$. To do that, we state that the directional derivative $f'(0; l)$ of $f(\cdot)$ exists for every $l \in S_1$ by (ii). We need to show that

- 1) the directional derivative $f'_l(0; \eta)$ of the auxiliary function $f_l(\cdot)$ exists for every $\eta = \pm 1$
- 2) the directional derivatives $f'(0; l)$ and $f'_l(0; \eta)$ are bounded by M for every $l \in S_1$ and $\eta = \pm 1$, i.e.

$$|f'(0; l)| \leq M \quad (l \in S_1), \max\{|f'_l(0; 1)|, |f'_l(0; -1)|\} \leq M.$$

We first have a closer look at the function, considering its values on different areas in \mathbb{R}^2 . It is not difficult to see that

$$f(x, y) = \begin{cases} |x|, & xy \leq 0 \quad (\text{area } A) \\ |x| - |y|, & \frac{x}{y} \geq 1, y \neq 0 \quad (\text{area } B) \\ \frac{|y|}{k} - |x|, & \frac{x}{y} \in \left[\frac{1}{2} \left(\frac{1}{k} + \frac{1}{k+1} \right), \frac{1}{k} \right], k \in \mathbb{N} \quad (\text{area } C_k) \\ |x| - \frac{|y|}{k+1}, & \frac{x}{y} \in \left[\frac{1}{k+1}, \frac{1}{2} \left(\frac{1}{k} + \frac{1}{k+1} \right) \right], k \in \mathbb{N} \quad (\text{area } D_k). \end{cases}$$

Since $f'(0; l) = f(l)$ by (iii), the absolute value of the directional derivative is bounded by M .

The areas A and B and the areas C_k and D_k for several initial values of k are shown on Figure 3.

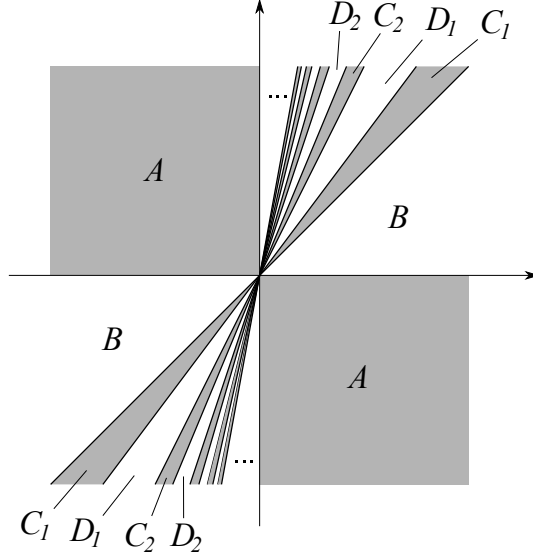


Figure 3: Regions on which the function f from Example 5.11 is linear

Observe that the function f is piecewise linear in a neighbourhood of every point except for the line $H = \{0\} \times \mathbb{R}$. Therefore, its directional derivative exists and is bounded by the Lipschitz constant of f everywhere except for H , and hence the restriction of f to the relevant affine subspaces is also defined and is bounded by the Lipschitz constant everywhere on $\mathbb{R}^2 \setminus H$. It remains to show that the restriction f_l of f is directionally differentiable for $l \in S_1 \cap H$. Observe that $S_1 \cap H = \{(0, 1), (0, -1)\}$. For $l^+ = (0, 1)$ and $l^- = (0, -1)$ we have

$$f_{l^+}(t) = f(t, 1) = \inf_{k \in \mathbb{N}} \left| t - \frac{1}{k} \right|, \quad f_{l^-}(t) = f(t, -1) = \inf_{k \in \mathbb{N}} \left| t + \frac{1}{k} \right| = f_{l^+}(-t).$$

It was shown in [5, Example 3.5] that the function f_{l^+} is directionally differentiable with $\max_{v \in \{-1, 1\}} |f'_{l^+}(0; v)| \leq 1 < L$. We have hence shown that the function f is M -directed subdifferentiable.

Acknowledgements

Partially supported by The Hermann Minkowski Center for Geometry at Tel Aviv University, Tel Aviv, Israel and by the University of Ballarat ‘Self-sustaining Regions Research and Innovation Initiative’, an Australian Government Collaborative Research Network (CRN).

We would like to acknowledge the discussions with Jeffrey C. H. Pang starting in October 2009 about Lipschitz definable functions. During the communications he suggested a construction to illustrate the existence of Lipschitz functions which are not quasidifferentiable. This motivated us to consider Example 5.11 which is based on a well-known one-dimensional example of a function that is quasidifferentiable but not a DC function (see [5, Example 3.5]). The authors are also grateful to Aris Daniilidis and Dmitriy Drusvyatskiy for a helpful discussion on Lipschitz definable/semi-algebraic functions.

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